

Kink-antikink collisions in the ϕ^6 model

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We study kink-antikink collisions in the one-dimensional non-integrable scalar ϕ^6 model. Although the single-kink solutions for this model do not possess an internal vibrational mode, our simulations reveal a resonant scattering structure, thereby providing a counterexample to the standard belief that existence of such a mode is a necessary condition for multi-bounce resonances in general kink-antikink collisions. We investigate the two-bounce windows in detail, and present evidence that this structure is caused by existence of bound states in the spectrum of small oscillations about a combined kink-antikink configuration.

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Introduction. Solitary wave solutions in non-linear field theories have been studied for many decades. Perhaps the simplest examples are the kink solutions which appear in models in one spatial dimension with a potential with two or more degenerate minima. These models have applications in condensed matter physics [1], field theory [2, 3] and as simplified models of domain wall collisions in cosmology [4–6]. The double well case is exemplified by the nonintegrable ϕ^4 model. A particularly striking feature of this theory is the intricate structure of kink-antikink scattering, first observed numerically in the 1970s [7] and much studied thereafter [8–13]. For initial velocities above a critical value $v_c \approx 0.2598$ the two incident waves always escape to infinity after collision, with the emission of some radiation. Below v_c , the incident waves generically become trapped, but there is also a complicated pattern of narrow ‘resonance windows’ within which they are again able to escape to infinity.

The accepted explanation of the appearance of these windows, both in this model and in similar theories such as the parametrically modified sine-Gordon model [10], is that they are related to a reversible exchange of energy between the translational and vibrational modes of the individual kinks [9]. At the initial impact, some kinetic energy is transferred into internal ‘shape’ modes of the kink and antikink. They then separate and propagate almost independently, but for initial velocities less than v_c they no longer have enough translational energy to escape their mutual attraction, and so they return and collide a second time. At this point some of the energy stored in the shape modes can be returned to the translational modes, tipping the energy balance back again and allowing the kink and antikink to escape to infinity, provided that there is an appropriate resonance between the interval between the two collisions, and the period of the internal modes. More generally, sufficient energy might be returned to the translational modes after three or more kink-antikink collisions, leading to an intricate nested structure of resonance windows, a pic-

ture which has been confirmed by both numerical and analytical studies [9–13].

For this mechanism to work, the kink and antikink must each support at least one internal vibrational mode, within which energy can be stored prior to being transferred back to the translational modes after a number of bounces. This has led to the commonly-held belief, stated for example in [12], that the existence of an internal kink mode is a *necessary* condition for the appearance of resonance windows. The example of the parametrically modified sine-Gordon model lends further support to this view: depending on the value of a parameter, kinks and antikinks do or do not possess an internal mode; correspondingly, resonance windows do or do not appear [10]. Resonances have also been observed in vector soliton collisions [14, 15] and in the scattering of kinks on impurities [16], but again, the mechanism always relies on the presence of a localised internal mode, either of a single kink or of an impurity, or both.

In the present Letter we revisit this question in the context of kink-antikink scattering in the ϕ^6 model [17]. The potentials mentioned in [10] include a general polynomial form, but the ϕ^6 model has a number of features whose relevance to resonant scattering have not been appreciated to date, even in the recent work [18]. As for the ϕ^4 model, it is non-integrable, but, as for the modified sine-Gordon model in the regime of no resonant scattering, a single ϕ^6 kink does not have an internal oscillatory mode [17]. Nevertheless, certain kink-antikink collisions exhibit resonant scattering, thus providing a counterexample to the standard lore. We elucidate the mechanism for this process, finding that the underlying reason is a reversible transfer of kinetic energy from the kinks to a collective bound state trapped by the $\bar{K}K$ pair.

The model. The one-dimensional ϕ^6 theory can be defined by the rescaled Lagrangian density [17]

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} \phi^2 (\phi^2 - 1)^2. \quad (1)$$

The model has three vacua $\phi_v \in \{-1, 0, 1\}$. Static

kinks and antikinks, interpolating between neighbouring vacua, can be found from the one-kink solution $\phi_K(x) \equiv \phi_{(0,1)}(x) = \sqrt{(1+\tanh x)/2}$ using the discrete symmetries of the model under $\phi \rightarrow -\phi$ and/or $x \rightarrow -x$, so $\phi_{\bar{K}}(x) \equiv \phi_{(1,0)}(x) = \phi_K(-x)$, $\phi_{(0,-1)}(x) = -\phi_K(x)$ and $\phi_{(-1,0)}(x) = -\phi_K(-x)$. These all have mass $M = 1/4$.

The model also possesses perturbative meson states, fluctuations about the vacua $\phi = 0, \pm 1$. Fluctuations around the static solutions $\phi_s(x)$ which interpolate between these vacua can be treated by setting $\phi(x, t) = \phi_s(x) + \eta(x)e^{i\omega t}$. The linearised field equations are $-\eta_{xx} + U(x)\eta = \omega^2\eta$ where the potential $U(x)$ is

$$U(x) = 15\phi_s^4 - 12\phi_s^2 + 1. \quad (2)$$

For an isolated kink there are no localised solutions to this equation beyond the usual translational zero mode, reflecting the absence of internal oscillatory modes. The states of the continuum spectrum can be written in terms of hypergeometric functions [17].

Numerical results. The initial conditions we took correspond to a widely separated kink-antikink pair propagating towards a collision point. To find a numerical solution of the PDE describing the evolution of the system, we used a pseudo-spectral method on a grid containing 2048 nodes with periodic boundary conditions. For the time stepping function we used a symplectic (or geometric) integrator of 8th order to ensure that the energy is conserved. The time and the spatial steps were $\delta x = 0.25$ and $\delta t = 0.025$. To check numerical stability we repeated selected calculations with $\delta t = 0.05$ and $\delta t = 0.0125$.

Since the model (1) contains two distinct classes of vacua, 0 and ± 1 , kink-antikink collisions in the sectors built on the $\phi_v = 0$ and the $\phi_v = \pm 1$ vacua have to be analysed separately. In the former case the initial configuration, which we denote as $K\bar{K}$ or $(0, 1) + (1, 0)$, can be taken as a superposition $\phi(x) = \phi_K(x+a) + \phi_{\bar{K}}(x-a) - 1$ where $a > 0$ is the initial (half-)separation parameter, which we set equal to 12.5; in the latter case we take $\phi(x) = \phi_{\bar{K}}(x+a) + \phi_K(x-a)$ and denote this as $\bar{K}K$ or $(1, 0) + (0, 1)$. In neither case are there internal vibrational modes bound to the individual kinks. By the standard picture, we would therefore predict that neither should exhibit any resonance structure.

In the case of $(0, 1) + (1, 0)$ $K\bar{K}$ collisions, this is indeed what we found (Figs. 1(a), (b)). For $v < v_{cp} \approx 0.289$ the pair always become trapped, while for $v > v_{cp}$ the collision yields a reflected pair of solitons together with some radiation: $(0, 1) + (1, 0) \rightarrow (0, -1) + (-1, 0)$. However the $(1, 0) + (0, 1)$ $\bar{K}K$ collision reveals a very different picture, illustrated in Figs. 1(c), (d) and (e). An intricate pattern of escape windows can be seen, up to a critical velocity $v_{cr} \approx 0.0457$, after which the kinks always have enough energy to separate. This is very similar to the behaviour of the ϕ^4 model, even though there are no internal modes of the scattering kinks into which kinetic energy can be

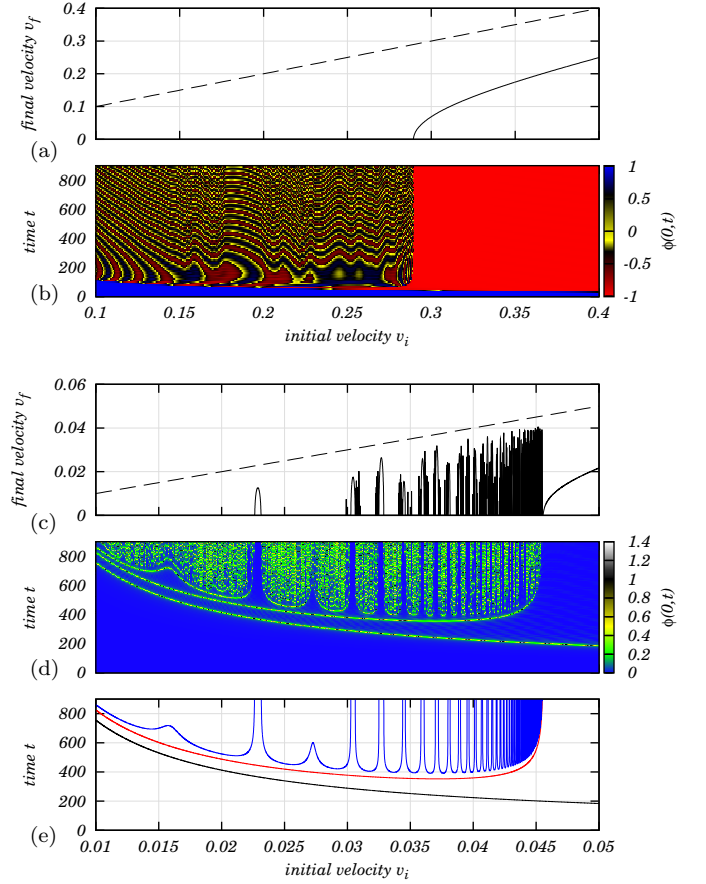


Figure 1. $\{K, \bar{K}\}$ collisions in the $(0, 1) + (1, 0)$ ((a), (b)) and $(1, 0) + (0, 1)$ sectors ((c), (d), (e)). Plots (a) and (c) show the fitted final velocity of the kink as a function of initial velocity, with the dotted line indicating the result for a purely elastic collision, while (b) and (d) depict the field values measured at the collision centre. The final plot, (e), shows the times to the first, second and third kink-antikink collisions in the $(1, 0) + (0, 1)$ sector.

transferred. Thus, we have to look for another explanation of our results.

The mechanism. The spectrum of linear perturbations around a single ϕ^6 kink differs from that for the ϕ^4 kink in that the potential $U(x)$ is not symmetrical with respect to reflections $x \rightarrow -x$. As a result the potential for a kink-antikink pair depends on the order in which they appear on the line: a well-separated $K\bar{K}$ pair (Fig. 2(a)) has a raised central plateau, while an equally separated $\bar{K}K$ pair (Fig. 2(b)) has instead a wide central well. If the velocities of the kinks are relatively small, we expect – and will justify later – that the adiabatic approximation can be used to find the spectrum of small fluctuations about these two configurations, using Eq. (2) for the appropriate choices of ϕ_s . Fig. 3(b) shows our numerical results for the $(1, 0) + (0, 1)$ $\bar{K}K$ pair as a function of separation parameter a . The two lowest states are quasizero modes, rapidly approaching zero from opposite sides as the separation grows. On top of these is a tower

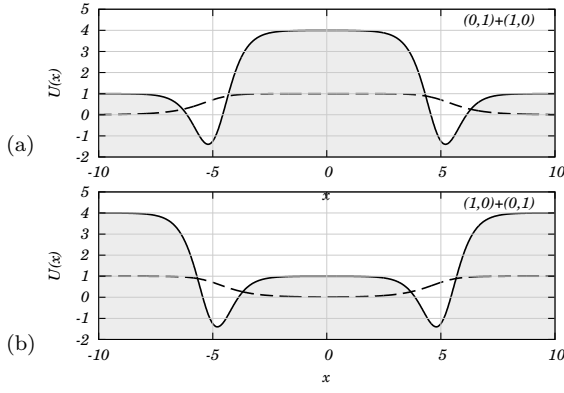


Figure 2. The potential $U(x)$ (solid lines) for linear perturbations about the composite $(0,1) + (1,0)$ $K\bar{K}$ (a) and $(1,0) + (0,1)$ $\bar{K}K$ (b) configurations (dashed lines).

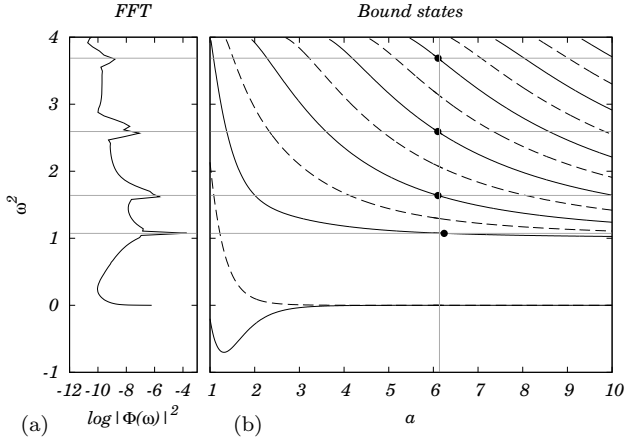


Figure 3. (a) Peaks of the Fourier transform of the field at the origin in the two-bounce window for $v = 0.04548$, vs. (b) even (solid) and odd (dashed) bound states for the $(1,0) + (0,1)$ configuration, as a function of the half-separation a .

of meson states with separation-dependent energies. The analogous plot for the $(0,1) + (1,0)$ $K\bar{K}$ pair, by contrast, shows only the two quazero modes.

So, a new feature of $\bar{K}K$ collisions in the ϕ^6 model is that although there are no internal modes of the individual kinks, energy can be stored after an initial impact in the trapped meson states of the composite $(1,0) + (0,1)$ configuration. This opens the possibility that, for collisions with $v_i < v_{cr}$ satisfying a suitable resonance condition, this energy might be returned to the translational modes on a subsequent recollision, thereby allowing the kink and antikink to return to infinity.

To support this picture, we first assign a ‘bounce number’ to each window, equal to the number of collisions that the kink and antikink undergo before their final escape to infinity. Within each window the bounce number is constant. The first two-bounce windows, discernible on Fig. 1(e) as the intervals of velocity within which the time to the third collision diverges, are centred at $v_i \approx 0.0228$, followed by a ‘false’ window at $v_i \approx 0.0273$, then a third at $v_i \approx 0.0303$, and so on. The regions near to these

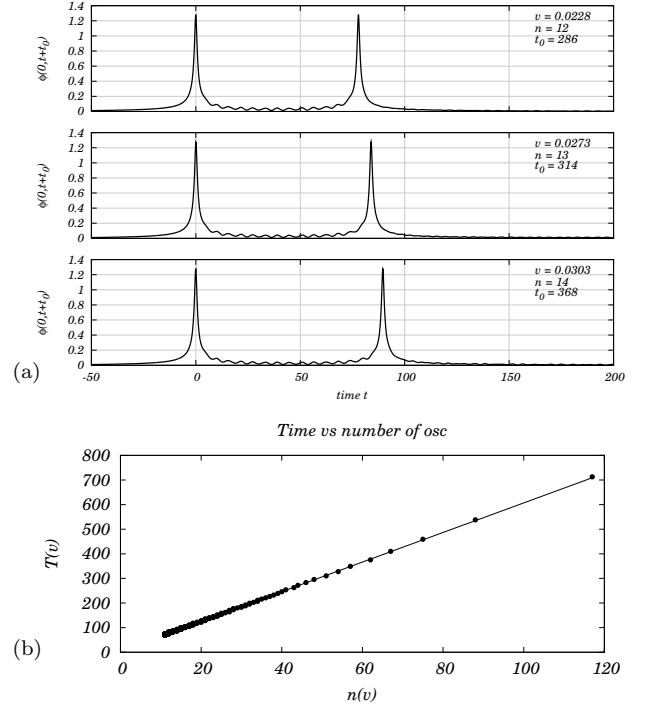


Figure 4. (a) Field value at the collision centre as a function of time from the first collision for three consecutive two-bounce windows. (b) The time between the two collisions in the two-bounce windows vs. the number of oscillations.

windows exhibit nested structures of higher-bounce windows, similar to those seen in kink-antikink interactions in the ϕ^4 model [11, 12].

Our analysis implies that each two-bounce window should be associated with a distinct integer n , the number of collective mode oscillations between its two collisions. In Fig. 4(a) we illustrate this for the three windows just discussed. It is easily checked that the number of oscillations of $\phi(0, t)$ between the two collisions increases by one from one plot to the next. Furthermore, our numerical results clearly indicate that these oscillations are those of the lowest collective mode of the $\bar{K}K$ system. In the two-bounce windows we considered, we observed maximal kink-antikink separations between bounces up to $R \equiv 2a \approx 12$. Fig. 3(a) shows the Fourier transform of the value of the field at the collision centre between the times of the first two bounces for $v_i = 0.04548$. The time between these bounces is $T \approx 660$, during most of which the separation between the kinks is close to 12. The Fourier transform shows four peaks at frequencies 1.045, 1.28, 1.61, 1.92, with the strength of the second peak about one order smaller than that of the first, and the higher ones strongly suppressed. The positions of these peaks relative to the energies of the first 4 parity-even modes (Fig. 3(a); note that the symmetry of the initial condition rules out any coupling to parity-odd modes) are consistent with a half-separation between the kink and antikink of $a \approx 6.13$, in good agreement with

the observed value. This also lends support to our use of the adiabatic approximation in the analysis of fluctuations about the kink-antikink configuration.

In Fig. 4(b) we show the dependence of the times $T(v)$ between the two collisions in the two-bounce windows on the number $n(v)$ of oscillations in between these collisions. The results are a good fit to the linear relation $\omega T = 2\pi n + \delta$, where $\omega = 1.0452$, the same as the frequency of the lowest collective mode at $2a \approx 12$ and the value found from the Fourier transform at $v_i = 0.04548$, and $\delta = 6.0277$. Requiring δ to lie between 0 and 2π , as here, fixes the number n of collective mode oscillations assigned to the three windows shown in Fig. 4(a) to be 12, 13 and 14 respectively, while the initial velocity $v_i = 0.04548$ used for the Fourier transform plot is very close to the $n = 109$ two-bounce window.

A linear fit with this value of ω is exactly as predicted by a resonance mechanism driven by the first mode, and closely resembles similar relations in the ϕ^4 model [9], although the detailed mechanism here is different. In particular, in resonant ϕ^4 scattering, energy is exponentially localised on the kink and antikink, while in the ϕ^6 theory this energy is stored in an extended meson state residing in the potential well formed between the kink and antikink. This results in a small amount of radiation pressure on the kinks after their initial impact, which we observed for $v_i = 0.05$, just larger than v_c , in a small (of order 10^{-6}) acceleration of the bouncing kinks away from each other, when measured at times $t \approx 300$ after collision so that the kinks had become well separated. A corresponding simulation in ϕ^4 theory showed that any such effect is at least 10^4 times weaker there.

Nevertheless, we can pursue the similarity between the two models by considering the asymptotic attractive forces between kink and antikink in the absence of radiation pressure, which are $F(a) \sim 2e^{-R}$ for the $\bar{K}K$ pair, and $F(a) \sim 2e^{-2R}$ for $K\bar{K}$ pair, where $R = 2a$ is the kink-antikink separation. The arguments of [9] would then predict $T(v) \propto (v_{cr}^2 - v_i^2)^{-\alpha}$ for impact velocities v_i just below v_{cr} , with $\alpha = 0.5$. Our fits based on this functional form favour a smaller value for α , in the range $0.39 - 0.45$. It would be interesting to see whether a more refined treatment, taking radiation pressure into account, could account for these discrepancies.

Perhaps the most intriguing feature of resonant ϕ^6 scattering, though, is the ‘missing’ window at $n = 13$. For resonant ϕ^4 scattering, two-bounce windows are also missing, for $n < 3$, but once they set in they are found for all n , at least up to initial velocities very close to v_c [9]. By contrast, in ϕ^6 scattering we found the first two-bounce window at $n = 12$, then a false window at $n = 13$, and then true windows for all higher values of n that we examined. Even more remarkably, a similar structure is reproduced when looking at the three-bounce windows next to a given two-bounce window, and we suspect that this pattern will continue at all higher levels. It

is possible that an explanation for this behaviour will be found in a careful treatment of the higher modes of the bound-state spectrum, but it remains a major challenge to convert this idea into a robust set of predictions.

Conclusions. The intricate structure of ϕ^6 kink collisions overturns some previous beliefs about resonant kink scattering, and shows that it has wider relevance than previously thought. The new mechanism enabling resonances to occur is the formation of meson bound states in the potential well created in the space *between* the constituents of a suitably-ordered kink-antikink pair. The resulting windows are less regular than for ϕ^4 scattering, and exhibit gaps. For large kink-antikink separations we also saw an additional long-range interaction through the trapped meson field, visible as radiation pressure.

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